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Three-Degree Graph and Design of an Optimal Routing Algorithm

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ABSTRACT

This paper suggests a novel graph CG_n (Circle Graph) with fixed degree of three. The node address of the CG_n is expressed with a n-binary number, along with degree of three, $3 \times 2^{n-1}$ edges, and 2^n nodes. In this paper, we suggest shortest-path routing algorithm for message transmission, and prove the diameter of CG_n is 2^{n-2} based on the routing algorithm outcome. Our CG_n possess the node symmetry and Hamilton cycle-structure.

Keywords: Algorithm, Graph, Routing, Parallel path, Shortest path

INTRODUCTION

In recent years, with the explosive growth of data volumes due to the development of IT technology, the number of high-performance computers is increasing due to escalating demands in diverse domains such as Big data analysis and artificial intelligence technology. Likewise, high-performance computer technology that operates multiple processing devices is also expanding, including attempts to increase the efficiency of CPU-centered processing units, GPU, and many-core processors. High-performance computers are evolving throughout the rapidly changing era, offering satisfaction to new market demands and diverse requirements on application domains.

A parallel computer is a computer system that subdivides a given job into multiple tasks and processes the corresponding task by allocating them to hardware resources (*e.g.*, multiple cores) in a parallel fashion. Parallel computers are mainly divided into shared-memory multiprocessors and message passing computers. In a shared-memory multiprocessor system, the memory system directly affects the overall performance. The critical attribute of the message passing computer is the interconnection network, which refers to the connection structure between processors considering the locational property, which is one of the factors that determine the performance of a parallel processing system. Therefore, research on interconnection networks is considered fundamental in order to improve the performance and control the execution of parallel processing computers.

The network cost is a widely implemented metric to quantitatively assess the interconnection network, mathematically defined as a multiplication between the degree and the diameter where degree refers to the hardware cost and diameter denotes the software cost. In order to construct an efficient network structure with the minimum usage of network cost, the cardinality of both the degree and diameter should be reduced. However, the relationship between the degree and diameter shows a negative correlation, and this trade-off hinders the achievement of decreasing the network cost in a harmonious manner.

The Shortest Path problem refers to a problem of finding an optimal route in which the sum of weights of edges leads to the minimum value that connects the node u to node v in the network G where $(n, v) \in G$. The edge weight is the value of the edge connecting the existing nodes and could be both positive to negative numerical values. This concept offers a numerical representation of resources and factors that evaluate the network (*e.g.*, cost, distance, time, etc.), which can have positive or negative values. The shortest path problem in a weighted graph (*i.e.,* edges denoted with weights) computes the shortest path that reaches the destination node v from the initial starting node u. Diverse research was conducted to effectively locate the shortest path concerning this problem. In this paper, we propose a novel degree graph CG_n that presents a degree of three. We analyze the theoretical properties of this graph structure and the shortest path algorithm with respect to the parallel routing algorithmic perspective. Our n-dimensional CG_n indicates the node address through n-bit binary, having a degree of three, 2^n number of nodes, and 2^{n-2} diameter.

This paper is organized as follows. Section 2 explores the interconnection network and the types of constant-degree graphs. In section 3, we define the novel interconnection network CG_n and analyze theoretical properties of the suggested graph and its degree, followed by presenting the routing algorithm. Finally, we conclude our works in section 4.

Related Works

In a multiprocessor system, a multiprocessor interconnection network is defined as the network connection to support communication between each existing processor that composes the target system. An interconnection network can be represented as an undirected graph, with each processor as a node and the communication channels between the processors as edges. If a channel exists between two processors, an edge is placed between those entities. An edge is an undirected edge that can transmit data in both directions. The interconnection network of parallel computers is represented as $G = (V,\ E)$ by an undirected graph G. $V(G)$ is a set of nodes inside the graph G, that is, $V(G) = \{0,\ 1,\ 2,\ \cdots,\ N-1\}$, and $E(G)$ is a set of edges composing the G. The edge $e(v, w) \in E(G)$ is an arbitrary connection between the nodes v and w where $(v, w) \in V(G)$, and the necessary and sufficient condition for the existence of $e(v, w)$ is a communication channel between the nodes v and w. A network with a relatively small diameter compared to the number of nodes have a short distance between nodes, but it becomes arduous to design the hard components as the number of nodes escalates.

Constant Degree Graph

The interconnection networks can be classified into the following three types based on the number of nodes: mesh type with $n(V(G)) = k \times n$, a hypercube

Table 1. Constant degree graph′.

Interconnection Network	Number of Nodes	Degree	Diameter	Network Cost
Mesh	$k \times n$	4	$2\sqrt{n}$	$O(8\sqrt{n})$
Honeycomb	$6t^2$	3	$1.63\sqrt{n}$	$O(4.9\sqrt{n})$
Torus	$k \times n$	4	$\sqrt{n}$	$O(4\sqrt{n})$
SEP_n	$n!$	3	$\frac{1}{8}(9n^2 - 22n + 24)$	$O(\frac{27}{8}n^2)$
$NSEP_n$	$n!$	4	$\frac{2}{3}n^2 - \frac{3}{2}n + 1$	$O(\frac{8}{3}n^2)$
CG_n	2^n	3	2^{n-2}	$O(3 \times 2^{n-2})$

type with $n\,V(G)) = 2^n$, and a star graph type with $n\,V(G)) = n!$. The mesh type is a well-known graph structure that has been widely utilized as a planar graph, which has been commercialized in various systems.

The m-dimensional mesh $M_m(N)$ consists of $n\,V(G)) = N^m$ and $n\,(E\,(G)) = mN^m - mN^{m-1}$. The address of each node is expressed with an m-dimensional vector, and when the addresses of any two nodes differ by 1 in one dimension, there is an edge between them. The advantage of Low-dimensional meshes is that they are straightforward to design, it is considered a promising scheme that has been widely implemented to specifically designs the network topology for parallel processing computers. Higher-dimensional meshes have smaller diameters and larger bipartite widths. In addition, they tend to perform the parallel algorithms with rapid pace, but requires a high cost. Hexagonal Mesh, Toroidal Mesh, Diagonal Mesh, Honeycomb Mesh, and Torus are proposed as an alternative structure that shows to have improved general lattice-structured mesh diameter. Furthermore, the Shuffle-Exchange Permutation (SEP) was asserted based on permutation groups, which has the advantage of easy conductivity of graph-based simulation (e.g., Cayley graph). SEP enables to efficiently operate the algorithm with minimal transitions in given new graphs, also is a regular network with a degree of three. Moreover, an NSEP (New-SEP) with a degree of four with enhanced SEP diameter and network cost have been proposed.

CG_n GRAPH DESIGN AND ANALYSIS

The Graph is formulated based on the Vertex (Node) and Edge. Let node of graph CG_n be S, where the node is expressed with binary n-bit, and the total number of node $n\,\bigcup_{\forall i} S_i) = 2^n$. The address of the node S in the CG_n graph is as follows.

$S = (S_n\,S_{n-1}\,S_{n-2}\,...S_i...\,S_3\,S_2\,S_1),\ 1 \leq\ i\ \leq\ n,\ i\in \mathbb{N}$

Definition 1. The three edge types connecting to node S constituting the n-dimensional CG_n graph are as follows.

1. Increasing Edge (E_f): Connect a node that is 1 bit higher from the address of the node S.

$$\{(S\,+\,1) + 2^n\,\}\ \%\ 2^n$$

2. Decreasing Edge (E_b): Connect a node that is 1 bit lesser from the address of the node S.

$$\{(S - 1) + 2^n\} \% \ 2^n$$

3. Complement Edge (E_c): Connect a node that is a complement node of the most significant bit S_n from the address of the node S.

$$(S + 2^{n-1}) \% \ 2^n$$

Example 1. In the n-dimensional CG_n graph, when $n = 4$, the connection relationship of nodes is as follows.

In G, $n\,(V\,(G))= 2^4$ with $E\,(v_{\forall i}) = 3$. If the node S is 1 ($=0001_{(2)}$) where $x_{(2)}$ denotes the binary with $x \in \{0, 1\}$, we substitute a decimal value according to the graph definition, which leads E_f to direct 2 ($=0010_{(2)}$) and E_b to 0 ($= 0000_{(2)}$). The final branch (E_c) points to 9 ($=1001_{(2)}$) in which only the leftmost bit is complemented, and this operation is applied equally to all nodes, and the graph of 2^4 has the form as shown in Figure 1.

Definition 2. In CG_n, a cycle of length 2^n consisting only of increasing or decreasing edges is nominated as H_n cycle.

Example 2. When $n = 3$, the H_3 cycle is a subgraph that has the length is 8 ([Fig. 2]).

A cycle that includes all vertices in the connected graph G is called a Hamiltonian cycle. If the network has a Hamilton pass or a Hamilton cycle, a ring or a linear array can be easily implemented, thus it is used as a pipeline for parallel processing. Theorem 1 shows that the internal CG_n network has a Hamiltonian cycle.

Theorem 1. CG_n contains the Hamilton cycle.

Proof) Let S be an arbitrary node. By definition 2, there is always a cycle starting from S and returning to S by repeating increment or decrement operations. Therefore, CG_n has a Hamiltonian cycle.

Theorem 2. CG_n is a symmetric graph.

Proof) A symmetrical relationship proves a one-to-one relationship between adjacent nodes of each existing node when mapping two arbitrary nodes. Let U be an arbitrary departure node and V be a destination node ($0 \leq U,\ V \leq 2^n,\ U \neq V$). By definition 2, arbitrary two nodes of CG_n can be reached via increasing or decreasing edges. Therefore, U and V have the following relationship.

$$U + \alpha = V, \ (0 < \alpha < 2^n)$$

The adjacent nodes of node U and V are $\{U + 1,\ U - 1,\ (U + 2^n)\ \%2^n\}$, $\{V + 1, V - 1,\ (V + 2^n)\ \%2^n\}$ respectively. When mapping U to V, the neighbor node is $\{(U + \alpha) + 1,\ (U + \alpha) - 1,\ ((U + \alpha) + 2^n)\ \%2^n\}$, and for $U + \alpha = V$, we conclude that $\{V + 1, V - 1,\ (V + 2^n)\ \%2^n\}$. Therefore CG_n is a symmetric graph.

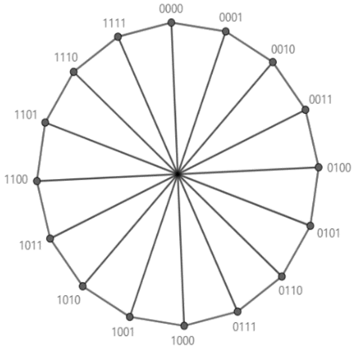

Figure 1: CG_4 Graph.

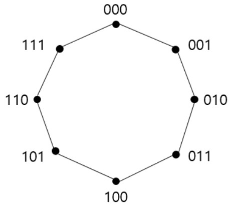

Figure 2: H_3 cycle.

Routing Algorithm

Routing is the process of efficiently and quickly determining the route of two points among one or multiple networks. We can obtain an effective solution to Routing problems in diverse domains via network techniques. A path is a route from one node to another through the edge of the graph. A practical routing algorithm should locate an optimal path, simultaneously simple. We introduce the routing algorithm and parallel path of the CG_n. Moreover, we prove the shortest path through the routing algorithm and parallel path algorithm.

Definition 3. D's location represents the set of destination nodes within the range when the starting node u is 0.

Our ultimate objective is to compute the optimal length, and the preliminary step is to determine the range where the destination node is located. Our approach is to divide the graph into two sections, and in the corresponding section, we compute the optimal length. Let the initial starting node is u, and we subdivide the section range of destination node v into Section A and Section B. Note that this study only divides and analyzes the right side of the graph while the routing algorithm is being processed.

Section A: $0 < D^{'}s\,location < 2^{n-2} + 1$
Section B: $2^{n-2} < D^{'}s\,location < 2^{n-1} + 1$

Example 3. In a graph where n is 3, the nodes in parentheses (001, 111), (010, 110), (011, 101) have the identical number of edges used to arrive at

Table 2. Notation definition.

Notation	Definition
S	Starting node (Initial node)
D	Destination node
Goal	Destination node
RN	Current node
FD, FD_s	Verify whether functions move_f() and move_b() were previously applied
D'	$2^n - D\ (D > 2^{n-1})$
move_f(a)	Apply edge E_f, Shift status by $((a+1)+2^n)\,\%2^n$
move_b(a)	Apply edge E_b, Shift status by $((a-1)+2^n)\,\%2^n$
move_c(a)	Apply edge E_c, Shift status by $\left(a+2^{n-1}\right)\%2^n$

the optimal length from the start node to the destination node when the initial node is 0.

(001, 111): $n\,(E\,(001,\ 111)) = 1$, (010, 110): $n\,(E\,(010,\ 110)) = 2$, (011, 101): $n\,(E\,(011,\ 101)) = 3$

The main reason for this is that the graph is symmetric by definition. That is, when obtaining the optimal length with (010) as the target node, the length of the edges leading to (110) enables the successful path.

Theorem 3. Let D be an arbitrary destination node of CG_n. The optimal length can be obtained through D' that satisfies $D + D' = 2^n$.

Proof) Let D be an arbitrary destination node of CG_n. With the formula: $D' = 2^n - D$, a node D' is being used as the destination node. The range is divided into two, and the operation selected first differs depending on the section in Sections A and B. Suppose the node belonging to Section A is the destination node. In that case, the increment operation is optimal, and if the node belonging to Section B is the destination node, the decrement operation is preferred. We verify this assertion through the routing algorithm.

Example 4. When $n = 4$, Section A, B is defined as follows.

Section A: Since it is a set of nodes greater than 0 and less than $2^{n-2} + 1$, it has a value of {1, 2, 3, 4} in decimal.

Section B: Since it is a set of vertices greater than 2^{n-2} and less than $2^{n-2} + 1$, it has a value of {5, 6, 7, 8} in decimal.

Definition 4. The terms used in the routing algorithm are defined as follows.

Example 5. When $n = 5$, if the initial node is 0 and the destination node is 1, the path is as follows.

Section A = {1, 2, 3, 4, 5, 6, 7, 8}

Section B = {9, 10, 11, 12, 13, 14, 15, 16}

Increment operation is conducted until it reaches the destination node since the destination node is an element of Section A.

$$0 \rightarrow 1$$

Algorithm 1. Routing Algorithm

```
Input: Starting node S, Destination node D
Output: int RN
Node move_f(node RN)
      return ((RN + 1) + 2^n)%2^n
Node move_b(node RN)
      return ((RN − 1) + 2^n)%2^n
Node move_f(node RN)
      return (RN + 2^(n−1))%2^n
Initialize int RN ← S, int (FD, FD_s) ← 0, int goal ← D
if (D < 2^(n−1)) then
   | FD ← 1
else
   | FD_s ← 1, D′ ← 2^n − D
   | D ← D′
if D ∈ Section A then
   | if FD > FD then
   |    | RN ← move_f(RN)
   |    | while (True) do
   |    |    | if RN = goal then break
   | else
   |    | RN ← move_b(RN)
   |    | while (True)
   |    |    | if RN = goal then
   |    |    |    | break
   |    | RN ← move_b(RN)
else if D ∈ Section B then
   | RN ← move_c(RN)
   | if FD > FD_s then
   |    | while (True) do
   |    |    | if RN = goal then
   |    |    |    | break
   |    |    | RN ← move_f(RN)
   | else
   |    | while (True) do
   |    |    | if RN = goal then
   |    |    |    | break
   |    |    | RN ← move_b(RN)
```

Example 6. When $n = 5$, if the initial node is 30 and the destination node is 15, the path is as follows.

Since destination node 30 is larger than 2^n, it utilizes the value of $D^{'}$.

$$D^{'} = 2^n - D = 32 - 30 = 2$$

Since $D^{'}$ is affiliated in Section A, it conducts decreasing operations until it reaches the destination node.

$$3 \to 2 \to 1 \to 0 \to 32 \to 31 \to 30$$

Example 7. When $n = 5$, if the initial node is 30 and the destination node is 15, the path is as follows.

Since destination node 15 is smaller than 2^n and affiliated in Section B, it conducts the incremental operation after the complement operation.

$$30 \to 14 \to 15$$

Optimal Length

In this section, we mathematically prove the formula that computes the Optimal Routing Path Length. In a graph with n bits, the optimal length from the single starting node S to the destination node D when the section is divided into two subparts according to the routing algorithm is as follows.

Section A: $\left|S - \left(\mathrm{D\ or\ } D'\right)\right|$

Section B: $|C(S) - (D\ or\ D')|$

In Fig. 1, we assume that the starting node S is 0 and the destination node D is 15. According to the formula $D' = 2^n - D$, D' becomes 1, and it is affiliated in Section A. Therefore, substituting (0-1), the absolute value is 1, which leads to the optimal length is also 1.

Theorem 4. The diameter of CG_n is 2^{n-2}.

Proof) According to Theorem 4, the distance is equivalent when the initial node is 0, and the destination node is 2^n. Note that the case of less than 2^n was divided into Section A and Section B. Since there is an edge E_c connecting the 2^{n-1} nodes with the farthest distance on the cycle H_n, the node with the farthest distance from node 0 is 2^{n-2} and 2^{n-1} in Section B. According to the algorithm, the 2^{n-2} applies the edge E_f and 2^{n-1} implements the edge E_b, followed by utilizing E_b to reach the destination node. In this case, the distance to 2^{n-2} and 2^{n-1} is equivalent to 2^{n-2}. Therefore, the diameter of CG_n is 2^{n-2}.

Example 8. When $n = 5$, let the initial node is 0, and the destination node is 7.

Since destination node 7 is associated with Section A, only increment operation is performed to reach the destination node.

$$0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7$$

Therefore, the diameter of CG_5 is 8.

CONCLUSION

In this paper, we proposed an interconnection network structure with a constant degree of three and analyzed its theoretical property. We defined the novel three-constant degree graph CG_n , also proposed the shortest path routing algorithm in propounded CG_n . The n-dimensional CG_n proposed in this paper represents a node address with n bits of a binary number, with a numerical degree of three, the 2^n number of nodes, and a diameter of 2^{n-2}. We show that the Hamilton cycle exists in internal CG_n , also has a maximum fault-tolerance. This novel graph topology and its property offer the viability of future implementation of a parallel processor system to enable high-performance computing operations, as well as providing insights and a reference to be applied in a real-world network system.

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